Theories on the Mechanics of Tires and Their Applications to Shimmy Analysis

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Some theories on tire mechanics and wheel shimmy are discussed and their results compared in an effort to clarify uncertainties as to the validity of the theories. Of particular interest is the comparison of the stretched string and the point contact theories of the mechanics of tires. Contrary to conclusions of some previous investigations, it is found that either of these fundamental, linear theories will predict the shimmy characteristics of wheeled systems if the parameters involved are properly chosen. Nonelastic effects and tire slippage can be and should be included in either theory if further improvement is desired. A simple but important correlation between certain of the parameters of these two basic tire mechanics theories is also demonstrated. The theories are compared with each other and with experimental data.

Nomenclature

A_m	= amplitude of mth oscillation peak
$B_0,\ldots,B_4$	= coefficients of characteristic equations
c	= tire yaw coefficient, rad/lb
c_1	= tire time constant, sec/rad
c_2	= second-order yaw stiffness coefficient, lb/rad^2
C_D	= equivalent viscous damping coefficient, in.-lb-sec/rad
c_L	= lateral tire damping coefficient, lb-sec/in.
c_v	= tire deflection proportionality coefficient, rad/in.
$\bar{c}$	= constant defined for Eqs. (20)
F_t	= lateral tire force acting on wheel, lb
h	= tire footprint half length, in.
$\bar{h}$	= constant defined for Eqs. (25), sec
I	= mass moment of inertia of tire test machine, lb-sec^2-in.
i	= complex unit, $(-1)^{1/2}$
k_1	= lateral tire stiffness, lb/in.
L	= trail length, distance from pivot to wheel center, in.
lc	= distance from pivot to system mass center, in.
m	= total system mass, lb-sec^2/in.
M_B	= moment due to pivot bearing friction, in.-lb
M_t	= moment on wheel due to tire deformation, in.-lb
s	= complex frequency of characteristic equations, rad/sec
t	= time, sec
U_1, U_2	= constants in stretched string theory, lb/in., in-lb
V	= rolling speed, in./sec
X, X_1, X_2	= lateral coordinates of motion, Figs. 2 and 4, in.
α	= variable defined for Eqs. (20)
$\Delta, \Delta_1, \Delta_2$	= lateral coordinates of tire deflection, Figs. 2 and 4, in.
μ	= coefficient of friction, tire ground
μ_1, μ_2, μ_3	= torsional stiffness constants of tire, in.-lb
σ	= tire stiffness constant/unit length, lb/in./in.
ψ	= angular deflection of wheel, rad
ψ_t	= yaw angle of tire footprint, rad
ω	= frequency of oscillation, rad/sec

I. Introduction

SEVERAL theories on the mechanics of flexible tires have been advanced. The motivation for the development of these theories has arisen, it would seem, from the desire to properly analyze automotive and aircraft wheel shimmy phenomena. Shimmy, whether it occurs as a simple vibratory nuisance or reaches the point of structural failure, is extremely undesirable and a proper understanding of the validity and applicability of the various theories is important. This paper attempts to point out some erroneous conclusions presented in previous papers on tire mechanics and shimmy in the hope of clarifying some fundamental misunderstandings surrounding the use of the appropriate mathematical models of tires used in shimmy analysis.

Once a set of coordinates has been chosen to describe a particular system, the governing equations of motion are derived by a straightforward application of standard techniques of dynamics. However, a difficulty arises when attempting to describe a satisfactory mathematical representation of the transmission of the forces and moments generated at the tire-ground contact region to the wheel. The importance of choosing the proper tire characteristics in predicting shimmy is discussed by Collins and Black.[1] The present paper is concerned with the more fundamental question of the choice of a particular theory of tire mechanics and its applicability and usefulness in determining the shimmy response of a system.

There are (at least) two approaches that may be taken to analyze a system for its shimmy stability characteristics. The most widely considered in the literature is the direct method where the complete differential equations of the system are solved by numerical integration or by the application of a stability criterion, depending upon the complexity of the system. A second technique which may be called the indirect method involves the determination of the mechanical admittance of the tire-ground constraint force. In this method, the frequency and phase response of the force and moment acting on the wheel because of the tire distortions are determined for specified wheel input harmonic motions. The impedances of the tire and the structure may then be compared so that regions of possible shimmy can be discerned. For the present paper, the direct method appears to be the most useful.

The major contribution of this paper is the comparison of two fundamental developments in the theory of the mechanics of tires: the stretched string theory of von Schlippe[2] modified herein and the point contact theory of Fromm,[3] de Carbon,[4] and Moreland.[5,6] There has been controversy as to the advantages and disadvantages of these methods (e.g., Smiley,[7] Segel,[8] Pacejka[9]); however, it is concluded here that either derivation can be used for qualitative, and in most cases quantitative, studies if the proper tire parameters are used. An advantage may be noted in the simplicity of the point contact theory. In order to compare these two basic tire theories, they are applied to a simple, freely castered, dynamic system which exhibits the shimmy phenomena and for which experimental data are available. A rather complete historical survey of contributions to the theories of tire mechanics may be found in the thesis of Pacejka.[9]

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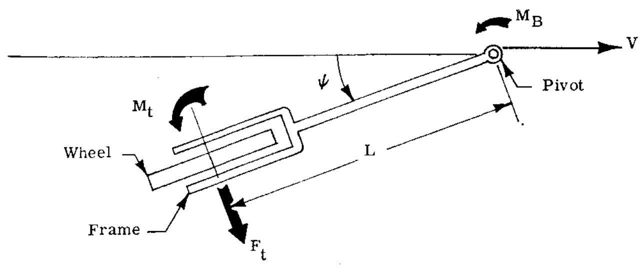

Fig. 1 Schematic diagram of simple dynamic test machine.

II. Dynamic Shimmy Model

In order that the tire theories considered in the subsequent portions of this paper can be compared for their usefulness in predicting shimmy, a mathematical model of a simple castered-wheel system is derived. The system of equations also describes an actual test machine which was developed to determine tire parameters from dynamic, oscillating conditions. A schematic diagram of the mathematical model is shown in Fig. 1. A more detailed description of the test machine and of the experimental data obtained from it, some of which is used herein, is found in Ref. 1. As can be observed from the plan view of Fig. 1, the machine is simply a frame which supports the wheel and is free to rotate about a pivot moving at a constant speed V. A given vertical load acts through the frame and presses the tire onto the roadway. This is the same model used extensively in the literature on shimmy.

The tire, constrained to roll without slipping, transmits the contact force F_t and moment M_t to the wheel rim. There will also be a moment M_B because of the frictional and damping losses at the pivot so that the moment equation about the pivot is

$$I\ddot{\psi} - LF_t - M_t - M_B = 0 \tag{1}$$

where the total moment of inertia about the pivot may be a function of the trail length L because of the design of the frame of the machine. The friction moment is expressed as an equivalent viscous damper,

$$M_B = -C_D\dot{\psi} \tag{2}$$

The equivalent viscous damping coefficient C_D was estimated from friction tests as a function of vertical load, trail length, and oscillation frequency of the tire parameter test machine of Ref. 1.

The tire force F_t and moment M_t are the subject of this paper and further discussion on the form of these functions follows. The trail length L and speed V are varied to obtain different stability characteristics.

III. Point Contact Theory

The simplest approach to the development of the functional relationships for the tire force and moment is probably that taken by Fromm,[3] de Carbon,[4] and Moreland.[5,6] The primary distinction between the point contact theory and the stretched string theory lies in the number of coordinates chosen to describe the deformation of the tire.

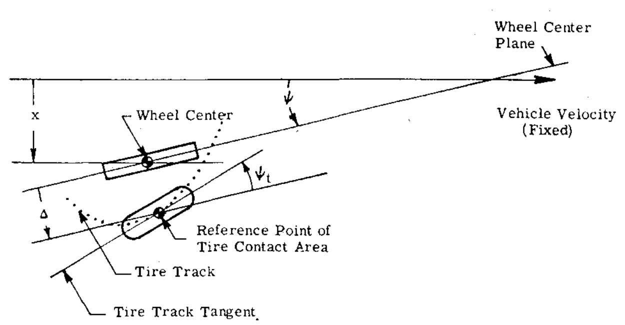

Fig. 2 Basic coordinates used to define tire deformation in the point contact tire theory.

Figure 2 shows the coordinates used for the point contact theory. The coordinate x is the transverse distance of the center of the wheel from the constant velocity line of the pivot (or vehicle) and the angle ψ is the angular rotation of the vertical plane passing through the wheel and normal to the axle. The deformation of the tire is indicated by the deflection Δ of the contact area reference point (say its geometric center) relative to the wheel center and the angle of twist ψ_t of the tire relative to the wheel plane.

Assuming that a side force acts on the wheel proportional to the tire deflection Δ and to the time rate of change of Δ, an effective linear spring-damper relation for the lateral force acting on the wheel results

$$F_t = k_1\Delta + c_L\dot{\Delta} \tag{3}$$

Similarly a linear torsional spring is assumed to represent the tire reaction to the twist ψ_t,

$$M_t = \mu_1\psi_t \tag{4}$$

This expression does not give a very good representation of the static twisting because of slippage on the outer edges of the contact region. However it is not as significant in its effect upon stability as the lateral tire stiffness k_1 or the tire yaw coefficient c, discussed below, and therefore Eq. (4) appears to be sufficient for preliminary dynamic studies.

An important factor influencing wheel shimmy when the point contact theory of tire mechanics is used is the effect of side force on the yaw angle ψ_t of the tire. If a wheel is forced to move so that its axle always remains in a fixed direction and a side force is applied to the axle (normal to the wheel plane), the wheel will move off in a fixed direction of yaw ψ_t. This effect is often noticed in an automobile as it travels down a roadway. A steady side wind will cause the automobile to drift to one side if a correction at the steering wheel is not made. This effect is well represented by the linear relation under steady state as $cF_t = \psi_t$.

Now since there must be some time delay between the application of the side force and the final steady-state yaw, it can be assumed that the inclusion of a term proportional to the rate of yaw $\dot{\psi}_t$ will improve the representation of the tire yaw. This is suggested by Moreland[6] and leads to a differential relationship between the yaw angle ψ_t and the lateral force F_t acting on the wheel

$$cF_t = \psi_t + c_1\dot{\psi}_t \tag{5}$$

The parameter c is the yaw coefficient for the tire and the parameter c_1 is called the tire time constant. Whether or not c_1 is a time constant in the classical sense for the general dynamical problem of shimmy can be and has been argued (see Ref. 7). However in the sense of the steady-state aperiodic motion discussed by Moreland[6] the idea of a time constant is quite obviously valid. Whether or not the terminology is appropriate is a moot point; what is important is the effect of the coefficient on the stability character of the system. It is shown that the "time constant" is useful and, in fact, necessary for a proper description of the system using the point contact theory.

Equations (3–5) give the force and moments acting on the wheel in terms of the tire deformation coordinates ψ_t and Δ. The system of governing equations of the motion is not yet complete as one other condition is needed to relate the three variables (ψ, Δ, ψ_t). This condition is provided by the constraint that the tire rolls upon the ground without slipping, which is essentially the classical nonholonomic constraint encountered in textbook rolling wheel problems. For small angles and tire deflections, this condition leads to the kinematic constraint

$$\dot{X} + \dot{\Delta} + V(\psi + \psi_t) = 0 \quad (6)$$

Equations (1 and 3–6) give a complete set of ordinary linear differential equations which may be solved to study the stability of the freely castered wheel system. In earlier studies other terms were sometimes included in the point contact equations such as de Carbon's turning coefficient of Ref. 4. Recent studies indicate, however, that the previous terms appear sufficient to adequately describe the shimmy phenomenon. In order to evaluate the validity of the point contact theory in determining shimmy stability, the tire equations developed in this section will be used with the dynamical system developed in Sec. II. The discussion below begins with a simplified system and then adds the various parameters to the equations in such a manner that their general effect can be observed.

Suppose the tire is simplified by considering it as a rigid member so that its flexibility is ignored as advised by Moreland.[5,6] This effect can be represented mathematically by $\mu_1, k_1 \to \infty$ and $c, M_t, \psi_t, \dot{\psi}_t, \Delta, \dot{\Delta} \to 0$. The force F_t is not zero but is determined by the governing equation of motion (1) while the motion itself is completely determined by the kinematic constraint Eq. (6) which, under these conditions has the solution,

$$\psi = \psi_0 \exp(-Vt/L) \quad (7)$$

The solution is exponential and completely stable for any configuration of the system as long as V, L are positive quantities. This result of course is not true for real systems as can be observed from the large quantities of experimental test data available in previous literature and that shown in Sec. VI of this paper. There are two likely possibilities for the discrepancy, first the dynamic model of the structure is not complete and second the tire cannot be assumed rigid. Moreland[6] assumes the first possibility advising that the tire can be conservatively assumed rigid. In Moreland's derivation the pivot is allowed to move laterally against a spring thereby adding a degree of freedom to the system of equations developed in Sec. II. When this Moreland system is analyzed, assuming a rigid tire, the criterion for stability is

$$ml_cL + LC_D/V - I > 0 \quad (8)$$

where l_c is the distance from the pivot to the mass center and m the total system mass. A curious result of this analysis is the absence of the lateral pivot spring constant from the stability criterion. This result led Moreland[5,6] to the conclusion that the fundamental cause of shimmy instability is because of the lateral flexibility of the structure or the pivot fixture in this case, and that the tire flexibility has an insignificant effect on the stability. However when the criterion in Eq. (8) is compared with experimental results of a test machine, such as that of Ref. 1 and Sec. VI of this paper, it gives very poor results. For instance in the case of tests on an 18 × 5.5 type VII aircraft tire, the previous inequality (8) predicts that, at any trail length less than 33 in., the system will be unstable, whereas the experimental data showed stability above 4 in. and very highly damped motion above 15 in. Therefore it appears that the destabilizing effect of the lateral structural flexibility is negligible for a very rigid, freely castered system and that the conclusion of Moreland is in error for such cases. The basic point to be gained here is that, in this system at least, the tire flexibilities and dynamics are more important factors than the structural flexibilities. It will be shown subsequently that when the tire is assumed flexible while the structure is assumed rigid the theoretical and experimental results can be reasonably compared. When considering typical aircraft and automotive systems both structural and tire flexibilities must be included for realistic results.

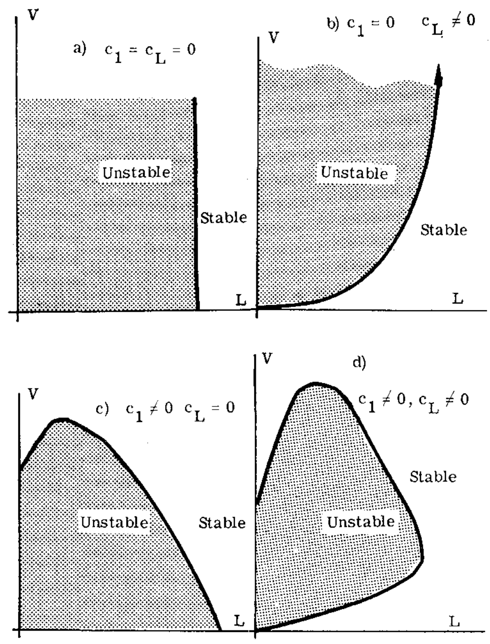

Fig. 3 Stability boundaries for structurally rigid free castering systems; $C_D = \mu_1 = 0$.

It is interesting and informative to consider the individual effects of the five tire parameters c, c_1, k_1, c_L, and μ_1. If $C_D = \mu_1 = c_1 = c_L = 0$, only c and k_1, the most fundamental parameters, effect the stability. The governing equations for the simple shimmy are of third order and lead to the de Carbon[4] stability criterion,

$$L > 1/(ck_1) \quad (9)$$

for stable oscillations. It is useful to visualize the stability regions and boundaries in the V, L plane as shown for the inequality of Eq. (9) in Fig. 3a. It is interesting to compare the simple result of Eq. (9) with that obtained from Eq. (8) and the experimental data previously discussed. Values of k_1 and c were measured with a vertical load of 775 lb and a tire pressure of 105 psi and found to be $k_1 = 1865$ lb/in. and $c = 1.34 \times 10^{-4}$ rad/lb, so that inequality (9) leads to $L > 4$ in. for stability which is in relatively close agreement with the experimentally observed value of about 3–4 in. at low speeds and is considerably better than the rigid tire result of inequality (8).

Now if the lateral damping c_L is included in the governing equations they remain third order, but the stability criterion becomes

$$c_Lk_1L^3/c + [(c_LL/c)^2 + Ik_1^2L - Ik_1/c]V > 0 \quad (10)$$

for stable oscillations. A sketch of this stability boundary is shown in Fig. 3b. Note the V is a single valued function of L and therefore the boundary cannot form a closed curve.

If the lateral damping c_L is again set equal to zero and the time constant c_1 is included the system becomes fourth order and has a stability boundary generally represented as in Fig. 3c which was sketched from the criterion,

$$V[c_1/L^2 - c_1^3/(cIL)] + 1/L - 1/(k_1cL^2) - c_1^2/(Ic) > 0 \quad (11)$$

for stability.

When both c_L and c_1 are retained the system remains fourth order, but the stability criterion becomes very unwieldy. However the boundary can now take on the closed form sketched in Fig. 3d which is often observed in experimental tests. It is probably more convenient to use a computer to find the roots of the characteristic equation for this system rather than plot the results of the stability criteria inequality,

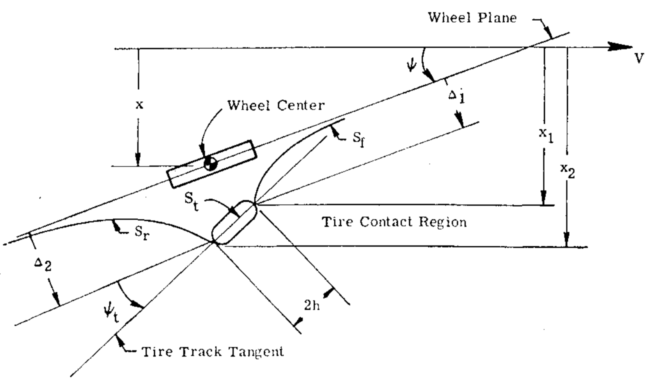

Fig. 4 Showing coordinates used in the stretched string tire model.

and for completeness the characteristic equation is presented below.

$$B_4S^4 + B_3S^3 + B_2S^2 + B_1S + B_0 = 0 \qquad (12)$$

where S is the complex frequency and the coefficients are

$$B_0 = V[L/c + \mu_1]/IL$$

$$B_1 = V[C_D/L + c_1/c + L/(Vc) + c_L/(k_1c) + c_L\mu_1/(Lk_1) + \mu_1/V]/I$$

$$B_2 = V/L + Lc_1/(Ik_1c) + \mu_1c_L/(Ik_1) + C_D/(ILk_1c) + C_DVc_L/(ILk_1) + Lc_1/(Ic) + Vc_Lc_1/(Ik_1c)$$

$$B_3 = [1 + Vc_Lc + L^2c_1c_L/I + C_Dc_1/I]/(Lk_1c)$$

$$B_4 = c_1/(Lk_1c)$$

Smiley[7] considered the use of the tire time constant and erroneously claimed it to be an invalid representation of the tire properties apparently because he used improper values for it in his studies. If the proper choice of values for c_1 is made the V,L stability boundary will close and provide a relatively accurate model of the freely castered system as is shown in Sec. VI of this paper. The torsional coefficient μ_1 does not appear to significantly effect the general character of the stability profiles, although the area of the unstable region does vary somewhat with various values of μ_1.

Changes in the tire parameters under various conditions, such as rolling speed, tire pressure, vibrational frequency, etc., should be considered since there is no reason to expect these parameters to be independent of such conditions. Past experience has shown that the yaw coefficient c is not greatly dependent upon speed, although the time constant c_1 is dependent upon it as would be expected. In the testing of Ref. 1, the time constant was assumed to be a function of speed only and not oscillation frequency which probably leads to some error in its determination.

IV. Stretched String Theory

The second fundamental approach to the theory of tire mechanics is that taken by von Schlippe.[2] Although most tires exhibit very nonelastic properties under deformation (especially torsional deformations) this theory, using the assumption of perfect elasticity, has met with a great deal of success. The tire is approximated by an elastic string, stretched about the outer edge of a wheel at the tire radius and attached to it by elastic springs. When there are no lateral or torsional deformations of the tire the string lies in the wheel plane. If the tire is under torsional and lateral deformation, it is assumed that the equivalent string takes the shape shown in Fig. 4. Details of the derivation for the tire force and moment as a function of the coordinates will not be given here as they are clearly presented in Refs. 2 and 7. However the fundamental assumptions made in the theory are summarized below. 1) The tire is assumed to be a string with a continuous linear elastic spring connecting it laterally to the wheel. The stiffness constant per unit length is σ which has the units lb/in./in. 2) At the leading point of the tire contact region, the lateral deflection of the tire relative to the wheel is related to the local angle of the tire track by the linear relation $\psi_{t(1)} = c_v\Delta_1$. The units of c_v are rad/in. 3) The string may be developed onto a flat surface as three separate curves shown in Fig. 4 which is a top view. The forward and aft curves (S_f,S_r) are represented as exponentials and the curve S_t is represented by a straight line.

With these basic assumptions, the tire force and moment are found by integration of the infinitesimal effects of the deformations all about the tire. The resulting equations are

$$F_t = U_1(\Delta_2 + \Delta_1)/2 \qquad (13a)$$

$$M_t = U_2(\Delta_2 - \Delta_1)/2h \qquad (13b)$$

where U_1 and U_2 are related to the coefficients σ, c_v and the tire contact length $(2h)$ as

$$U_1 = 2\sigma(h + 1/c_v) \qquad (14a)$$

$$U_2 = 2h\sigma[(h + 1/c_v)/c_v + h^2/3] \qquad (14b)$$

If the assumption 3 is valid, the quantities Δ_1 and Δ_2 may be written in terms of Δ and ψ_t used in the point contact theory so that

$$F_t = U_1\Delta \qquad (15a)$$

$$M_t = U_2\psi_t \qquad (15b)$$

Now the quantities Δ and ψ_t can be related directly to the track coordinates x_1 and x_2 for the simple dynamic system, as

$$\Delta = [(x_1 + x_2)/2] - X \qquad (16a)$$

$$\psi_t = [(x_2 - x_1)/2h] - \psi \qquad (16b)$$

which can be observed directly from Figs. 4 and 1.

Considering the simple dynamic system the variables x and ψ are related as $x = L\psi$. The kinematic constraint similar to Eq. (6) for the point contact theory is written here for the point of the leading edge of the contact region (x_1). This equation becomes

$$\dot{x}_1 + c_vVx_1 + V[1 + c_v(h - L)]\psi = 0 \qquad (17)$$

One other condition is needed to have a complete set of integrable equations because of the occurrence of x_2 in Eqs. (16a) and (16b). This is the equation specified by the assumption of pure rolling all along the line of contact S_t. Since the rear of the tire must roll across the same point as the front of the tire a time Δt later, where $\Delta t = 2h/V$, the delay condition must be satisfied:

$$x_2(t) = x_1(t - 2h/V) \qquad (18)$$

Equation (18) combined with Eq. (17) forms a difference differential equation for x_2 which leads to some difficulty in obtaining a complete solution. However, the mathematical complexities will be overlooked here and the problem of a solution will be attacked essentially as done in Ref. 2.

The governing equations defining the castered system are given by Eqs. (1, 2, and 16–18); however, one further addition is made for the inclusion of the damping effects of the tire. The lateral force Eq. (15) is extended to include the term $c_L\dot{\Delta}$ or $F_t = U_1\Delta + c_L\dot{\Delta}$.

The characteristic equation for this system is found by classical means and takes the following form where s is the complex frequency:

$$a_3s^3 + a_2s^2 + a_1s + a_0 + (as + b)\exp(-2hs/v) = 0 \qquad (19)$$

The frequency equations for the V,L stability boundaries are found by the substitution of the neutrally stable solution $s = i\omega$ and defining the variable α as $\alpha = 2h\omega/V$, the following equations result:

$$(I/2h\bar{c})\alpha\omega^2 - [(U_1L^2 + U_2)/2h\bar{c}]\alpha - [L^2c_Lc_v/\bar{c} + Lc_L/2 + C_Dc_v/\bar{c}]\omega - (Lc_L/2)\omega\cos\alpha + [(U_1Lh + U_2)/2h]\sin\alpha = 0 \quad (20a)$$

$$(Ic_v/\bar{c})\omega^2 + [(L^2c_L + C_D)/2h\bar{c}]\alpha\omega - [(U_1Lh - U)/2h + (U_1L^2 + U_2)c_v/\bar{c}] + - (Lc_L/2)\omega\sin\alpha - [(U_1Lh + U_2)/2h]\cos\alpha = 0 \quad (20b)$$

where $\bar{c} = 1 + c_v(h - L)$. The stability boundary is obtained by finding values of V and L which satisfy Eqs. (20).

It is of interest to compare the equations of this theory with those of the point contact theory. The essential difference lies in the use of the finite tire length $2h$ in the stretched string theory and the time constant c_1 in the point contact theory. The theories are equivalent if the footprint length h and the time constant c_1 are neglected. Letting $h \rightarrow 0$, the following results are observed: $x_2 \rightarrow x_1 \rightarrow x + \Delta$, $\psi_{t(1)} \rightarrow \psi_t$, and $c_v \rightarrow \psi_t/\Delta$, and if these conditions are substituted along with $x = L\psi$ into Eq. (17), Eq. (6) will result. If the time constant c_1 is also neglected the two theories give identical results.

Now the fundamental constants c_v, σ are directly related to the constants c, k_1, by considering the reduction of the equations for steady-state motion where $\dot{\psi}_t = \dot{\Delta} = 0$. The relation between U_1 and k_1 gives, for the von Schlippe constants,

$$U_1 = 2\sigma(h + 1/c_v) = k_1 \quad (21)$$

also from previous assumption 2 and Eqs. (3) and (5), it is deduced that

$$c_v = k_1c \quad (22)$$

combining Eqs. (21) and (22),

$$\sigma = k_1c_v/[2(1 + hc_v)] \quad (23)$$

so that, given h, the coefficients c_v, σ can be determined from k_1, c. The footprint length h can be determined from μ_1 using the relation (14b) as

$$U_2 = 2h\sigma[(h + 1/c_v)/c_v + h^2/3] = \mu_1 \quad (24)$$

however because of the lack of good data for μ_1, it is probably best to estimate h from static vertical loading tests and compute μ_1 from Eq. (24).

Reference 2 gives a discussion of the stability boundaries obtained from Eqs. (20) with $c_L = 0$ and $I = \text{const}$, so that it should not be necessary to repeat that here. It should be noted that as $h \rightarrow 0$, the stability criterion with Eq. (22) previously mentioned becomes identical to the inequality of Eq. (9) as would be expected. Also, it is worthwhile to note that, in general, there are an indefinite number of solutions for V and L from Eqs. (20). This is because of the appearance of the trigonometric functions which in turn are because of the time delay relation of Eq. (18). As $h \rightarrow 0$, this problem disappears and Eqs. (20) become standard polynomials, characteristic of the stability boundary equations of ordinary differential equations. One must be cautious in using these equations for large α (small V) as extraneous solutions may be uncovered for some system configurations.

V. Modifications and Extensions

Several additions and improvements have been offered in the literature to the tire theories as presented in Sec. III and IV and some pertinent ones are discussed in this section. Smiley[7] suggests that the time delay relation, Eq. (18), of the stretched string theory can be approximated quite well by representing it with a truncated Taylor's series expansion. This is equivalent to representing the exponential function in Eq. (19) by a truncated series expansion thereby eliminating the trigonometric terms in Eqs. (20). This also provides a convenience for the present analysis since the roots of Eq. (19) can be found directly from the resulting polynomial and therefore the amplitude ratios of the major oscillatory modes can be computed giving information not only of the location of the stability boundary but for the entire V, L plane. Therefore a more thorough comparison can be made with the point contact theory.

When the exponential is expanded to third order and Eq. (19) is rearranged, a fourth-order polynomial results where the coefficients are

$$B_4 = -a(\bar{h})^3/6$$
$$B_3 = I + [a - b\bar{h}/3]\bar{h}^2/2$$
$$B_2 = C_D + L^2c_L + IVc_v - [a - b\bar{h}/2]\bar{h} \quad (25)$$
$$B_1 = U_2 + L^2U_1 + VC_Dc_v + L^2Vc_vc_L + LV\bar{c}c_L/2 + a - b\bar{h}$$
$$B_0 = c_vVU_2 + c_vVL^2U_1 - U_2\bar{c}/\bar{h} + U_1VL\bar{c}/2 + b$$

where $a = L\bar{c}Vc_L/2$, $b = V\bar{c}(U_2/h + LU_1)/2$, and $\bar{h} = 2h/V$. These equations should be compared with Eq. (12) of the point contact theory.

It is imperative to note that the use of a stability criterion, such as Hurwitz' determinants, to investigate the stability of this system will, in general, give very poor or completely erroneous results. This is because of the possibility of false roots being introduced by the truncated series representation of the exponential. In fact, in determining the stability characteristics of the fourth-order equation previously mentioned, it can be observed that the coefficient B_4 is negative if $\bar{c}$ is positive so that with B_0 positive the system will be unstable under any other conditions. This result is in considerable error as the system may be quite stable under these conditions, for instance, if the damping is very large. In this particular case, the difficulty lies in the large positive false root resulting from truncation of the exponential at third order. If this root is neglected then the remaining three roots give a good approximation to the exact solution of Eq. (19).

Bergman[10] extends the stretched string theory to include slippage in portions of the tire-ground contact area where the interface shearing forces are greater than the critical adhesive value. The results of this theory can also be used with the point contact theory since there is no dependence upon the kinematic rolling condition and since the assumption is again made that the curve of the string is essentially a straight line in the region of contact. The same reduction is possible here that was used to obtain Eqs. (15). The form of the force and moment laws obtained by Bergman are essentially as shown below:

$$F_t = \psi_t/c - c_2\psi_t^2/\mu \quad (26a)$$

$$M_t = \mu_1\psi_t - \mu_2\psi_t^2/\mu + \mu_3\psi_t^3/\mu^2 \quad (26b)$$

where μ is the coefficient of friction between the tire and the ground. It can be observed from Eqs. (26) that the lower the friction coefficient the more the force and moment will deviate from the ideal linear laws assumed in both the stretched string and point contact theories. It appears that on normal roadway surfaces, the coefficient c_2/μ is small enough that the linear relation is sufficient for the representation of F_t, however, nearly all the static and steady-state data on M_t show a highly nonlinear characteristic. This can be observed, for instance, from the data of Refs. 10 and 11. Although the effects of the second- and third-order terms in Eq. (26b) are prominent in the static and steady-state yawed rolling of a tire, it appears that the linear relation may be successfully used for preliminary shimmy studies. The linear form is used in the studies of Sec. VI where it was found useful

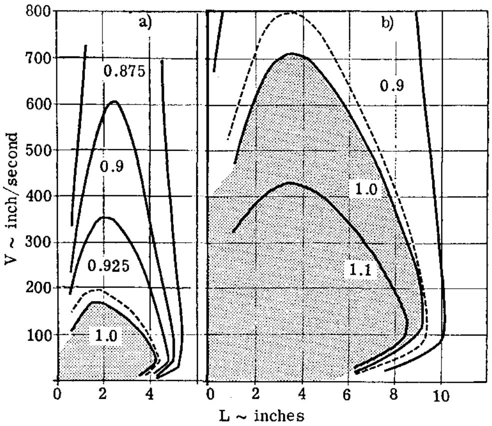

Fig. 5 Stability contours of stretched string theory; contours are constant amplitude ratios $|A_{n+1}/A_n|$; dashed curves - - - exact theory, solid —— approximation of Smiley; for 18 × 5.5 tire with vertical load of a) 775 lb and b) 3100 lb.

in obtaining good correlation between experimental and theoretical oscillation frequencies.

Pacejka[9] presents a generalized theoretical study of the mechanics of tires. His work, an extension of the stretched string theory, uses multiple parallel strings to simulate the width of the tire. He also includes elastic elements (profile elements) which are attached to the strings and situated between the strings and the road surface, thus, apparently more closely simulating the rubber covered cord of an actual tire. His considerations also include slippage of the contact area and are, in fact, primarily involved with the nonlinearities which arise from this slippage as well as from other mechanical elements. Pacejka indicates that the extension of the stretched string theory into a "beam" theory (replacing the elastic strings by elastic elements capable of supporting a bending moment) does not significantly improve the theory. He also considers and extends somewhat the work done by Segel[8] in the area of nonstationary shimmy characteristics.

It is important that tire theories be extended and refined and the theoretical aspects be improved in order that a better understanding of the mechanics of tires is attained. However, for engineering design purposes, there should no longer be a question of the general validity of these theories but only of the degree of experimental correlation desired of the mathematical model. For many cases, it may well be that a demonstration of a high degree of stability is all that is desired, and these cases it appears that the fundamental theories as presented in Secs. III or IV, with or without the modification suggested by Smiley of this section, are sufficient.

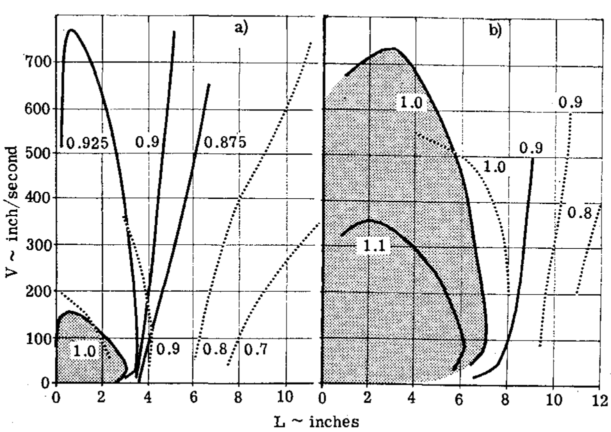

Fig. 6 Stability contours of the point contact theory (solid ——) and experimental data (dotted · · ·); contours are constant amplitude ratios $|A_{n+1}/A_n|$; for 18 x 5.5 tire with vertical load of a) 775 lb and b) 3100 lb.

Table 1 Parameters used in tire shimmy analysis

Tire	18 × 5.5	
F_z	775	3100
k_1	1865	1615
c	1.53×10^{-4}	0.75×10^{-4}
μ_1	1.16×10^4	3.98×10^4
c_L	12.6	8.9
c_1	$(0.04V + 2) \times 10^{-4}$	$0.011V \times 10^{-4}$
h	1.69	2.9

VI. Comparisons and Correlations

In this section, the fundamental theories will be compared with each other and with experimental data. The computations in this analysis are, for the most part, digital computer solutions of the characteristic equations developed in the earlier parts of this paper to represent the motion of the simple dynamic system. The test data used for comparison are from an 18 × 5.5 type VII, 14 ply rating tire loaded vertically with 775 lb and 3100 lb and inflated to 105 psi. Table 1 shows the tire parameters used to obtain the roots of the characteristic equations. In Table 1, the values of μ_1 were determined from Eq. (24) using the values of c and k_1 which were measured in the testing described in Ref. 1. The value of h was somewhat arbitrarily determined to be 75% of the static or standing tire half footprint length obtained from Ref. 11. The value of h should be reduced from that determined from static tests to allow for the outer edge slippage which will take place during rolling conditions.

Figure 5 shows the results of a study of the free dynamic response of the simple dynamic machine of Sec. II using the stretched string tire theory of Sec. IV and the modification by Smiley of Sec. V. The dotted curves represent the solution to the "exact" Eq. (20) and should be compared to the solid curve of amplitude ratio 1.0 of the approximate solution. The amplitude ratios are obtained from the real part of the prominent roots of the characteristic equation whose coefficients are given by Eqs. (25) and where the false root is ignored as suggested in Sec. V. The relationship for the ratio of adjacent peaks then becomes $|A_{m+1}/A_m| = \exp(n\pi/\omega)$ where the roots of the characteristic equation are $s = n \pm i\omega$.

For the cases shown in Fig. 5, it is observed that the approximation is very good, but in both loading conditions the instability region is inside that of the "exact" solution and therefore is not conservative although the difference is small.

The results for these conditions are also shown in Fig. 6 where the point contact theory of Sec. III is used. Again the contours are of constant amplitude ratios obtained as before but from the prominent roots of Eq. (12). Also shown are contours estimated from experimental data. Observation of Figs. 5 and 6 leaves little doubt of the similarity of the results of the two theories and of the relatively good comparison of either theory with experiment. Further comparisons are given in Fig. 7 where the frequencies of oscillation are plotted for the approximate stretched string theory, the point contact theory and from experiment. The approximate string theory, with amplitude ratio of one, gives frequency results so close to the exact that the frequencies for the exact equations are not presented.

In preparing Figs. 5–7, the available tire parameter information was used, and there was no attempt to match the results of the string and point contact theories. Therefore, the quantitative discrepancies between these theories can be reduced, if desired, by using modified values of the tire parameters. For instance, the footprint length $2h$ and the time constant c_1 can be varied from the values given in Table 1 so that the amplitude contours and the frequencies can be made to agree much more closely. Also, the peak of the instability region in Fig. 6a, for instance, may be moved toward larger values of L by reducing the value of μ_1, and, consequently, a

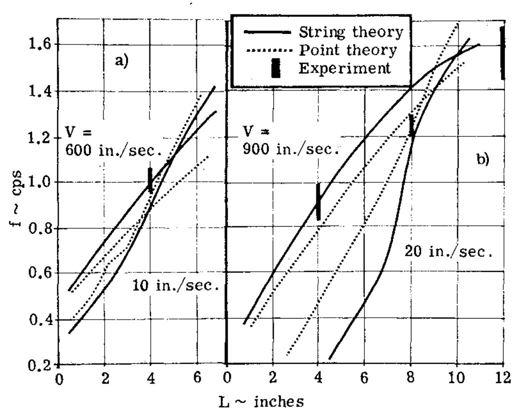

Fig. 7 Frequency results of stretched string theory, point contact theory and experiment; for 18 × 5.5 tire with vertical load of a) 775 lb and b) 3100 lb.

closer resemblance to Fig. 5a could be attained. The objective in the presentation of Figs. 5–7 was to show the results one can expect by taking measured parameters and applying them directly to both theories without further refinement. Tests were also performed on a 49 × 17 type VII tire and an analysis similar to the one previously performed. The results of this analysis showed a similar comparison between the exact and approximate string theory, the point contact theory and experimental data.

VII. Concluding Remarks

The problem of analyzing a wheeled system for shimmy requires the knowledge of the tire characteristics and a valid theory of how the forces of contact at the road surface are transmitted to the wheel. Either of the two basic linear tire mechanics theories discussed herein, appears adequate for qualitative and, in most cases, quantitative analysis. Although most tires show a highly nonelastic or hysteretic deformation character upon static loading, the linear point contact and stretched string theories will provide remarkably good information of the dynamic shimmy phenomenon when used in the proper manner. The point contact theory has some advantage over the string theory in its simplicity. Although the approximate string theory proposed by Smiley is simple, there is a feeling of uncertainty when using it because of the truncation of the infinite series representing the exponential function.

If it appears desirable to improve the results beyond that obtainable from the linear theories, the inclusion of tire slippage should be considered along with the hysteretic effects of the tire carcass. The difficulty in obtaining good theoretical-experimental correlation on tire twisting torques seems to be due mostly to the footprint slippage and variations in the friction coefficient.

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